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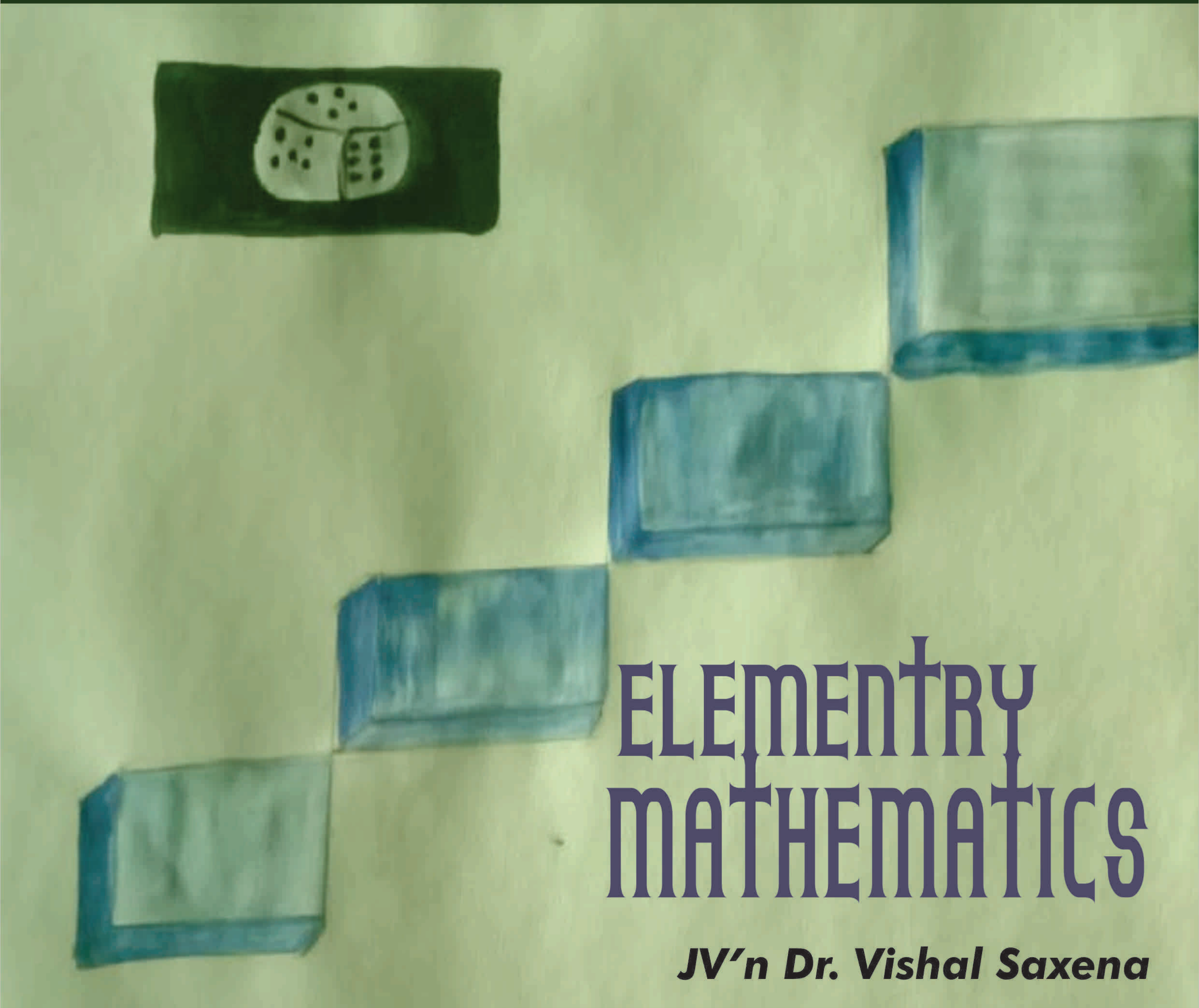

ELEMENTRY MATHEMATICS

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JAYOTI VIDYAPEETH WOMEN'S UNIVERSITY, JAIPUR

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Chapter 1. Matrices and Determinants

Matrix theory is an important part of mathematics. It is an area of active research and used by every mathematician and by many scientists working in different fields. Matrix theory is used to solve linear equations. A matrix represents a collection of numbers (real or complex) arranged in an order of rows and columns. The concept of determinant is based on matrix. Determinants are defined for square matrices only. The determinant of the matrix must be non singular for the system of equations to have a unique solution.

Matrix

A matrix is an rectangular arrangement of numbers (real or complex) in rows and columns. A matrix can be used to solve the systems of linear equations.

Examples:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 5 & 8 & 6 \\ 9 & 2 & 7 \end{bmatrix}$$

A matrix of order m×n is given by

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

In compact form $A = \left[a_{ij} \right]_{m \times n}$

Types of Matrices :

(i) **Null Matrix :** A matrix $A = \left[a_{ij} \right]_{m \times n}$ is called a null matrix if every element of matrix is zero.

Example: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(ii) **Square Matrix :** A matrix $A = \left[a_{ij}\right]_{m\times n}$ is called a square matrix if m=n.

(iii) **Diagonal Matrix :** A square matrix $A = \left[a_{ij}\right]_{m\times n}$ is called a diagonal matrix if all the non-diagonal elements are zero.

(iv) **Unit (Identity) Matrix :** A diagonal matrix $A = \left[a_{ij}\right]_{m\times n}$ is called a unit matrix if all the diagonal elements are unity.

(v) **Symmetric Matrix :** A square matrix $A = \left[a_{ij}\right]$ is called a symmetric matrix if $\left[a_{ij}\right] = \left[a_{ji}\right]$ for all I, j.

(vi) **Skew-Symmetric Matrix :** A square matrix $A = \left[a_{ij}\right]$ is called a skew-symmetric matrix if when $\left[a_{ij}\right] = -\left[a_{ji}\right]$ for all I, j.

(vii) **Orthogonal matrix :** A square matrix $A = \left[a_{ij}\right]$ is called a orthogonal matrix if $AA^T = I_n = A^TA$

(viii) **Idempotent matrix :** A square matrix $A = \left[a_{ij}\right]$ is called an Idempotent matrix if $A^2=A$.

(ix) **Involuntary matrix :** A square matrix $A = \left[a_{ij}\right]$ is called an Involuntary matrix if $A^2=I$

(x) **Transpose of Matrices :** The transpose of a matrix is determined by interchanging the rows for columns i.e. if matrix A = (a_{ij}) then transpose of A is:

A^T = (a_{ji}) where j and i are the column and row numbers respectively of matrix A.

Example:

$$A = \begin{pmatrix} 5 & 2 & 3 \\ 4 & 7 & 1 \\ 8 & 5 & 9 \end{pmatrix} \quad A^T = \begin{pmatrix} 5 & 4 & 8 \\ 2 & 7 & 5 \\ 3 & 1 & 9 \end{pmatrix}$$

For a square matrix, the transpose can be used to check if a matrix is symmetric or not(For a symmetric matrix $A = A^T$).

Algebra of Matrices

(i) Matrix Addition:

Two matrices A and B can be added if their orders are the same. Let the matrices A and B are

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix}$$

If A and B above are matrices of the same order, then the sum is determined by adding the corresponding elements a_{ij} and b_{ij} .

Here addition of A and B is given by:

$$A + B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 5 \\ 2 & 0 & 5 \end{pmatrix}$$

(ii) Matrix Subtraction:

If A and B above are matrices of the same order, then the determined is found by subtracting the corresponding elements a_{ij} and b_{ij}.

Here is an example of subtracting matrices.

$$A - B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

(iii) Matrix Multiplication:

The matrix multiplication can be performed when the number of columns of the first matrix is same as the number of rows of the second matrix.

Here is an example of matrix multiplication for two 2×2 matrices.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} (ae + bg) & (af + bh) \\ (ce + dg) & (cf + dh) \end{pmatrix}$$

Here is an example of matrix multiplication for two 3×3 matrices.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} j & k & l \\ m & n & o \\ p & q & r \end{pmatrix} = \begin{pmatrix} (aj+bm+cp) & (ak+bn+cq) & (al+bo+cr) \\ (dj+em+fp) & (dk+en+fq) & (dl+eo+fr) \\ (gj+hm+ip) & (gk+hn+iq) & (gl+ho+ir) \end{pmatrix}$$

The Determinant of a Matrix

Study of determinants is important in finding the inverse of a matrix and also in finding the solution systems of linear equations. In the following we assume we have a square matrix (m = n). The determinant of a matrix A will be denoted by det (A) or |A|. Initially the determinant of a 2×2 and 3×3 matrix are taken, then the n×n case will be explained.

Determinant of a 2×2 matrix:

Assuming A is an arbitrary 2×2 matrix A, where the elements are given by:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

then the determinant of this matrix is defined as:

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Determinant of a 3×3 matrix:

The determinant of a 3×3 matrix is complicated and is determined as follows (for this case assume A is an arbitrary 3×3 matrix A, where the elements are given below).

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

then the determinant of a this matrix is is defined as:

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinant of a n×n matrix:

For the general case, where A is an n×n matrix the determinant is given by:

$$\det(A) = |A| = a_{11}\alpha_{11} + a_{12}\alpha_{12} + \ldots + a_{1n}\alpha_{1n}$$

Here the coefficients α_{ij} are given by

$$\alpha_{ij} = (-1)^{i+j}\beta_{ij}$$

where β_{ij} is the determinant of the (n-1) × (n-1) matrix that is obtained by deleting row i and column j. This coefficient α_{ij} is known as the cofactor of a_{ij}.

The Inverse of a Matrix

Suppose we have a square matrix A and which is non-singular (i.e. det(A) does not equal zero), then there exists an n×n matrix A^{-1} which is called the inverse of A, such that this property holds:

$AA^{-1} = A^{-1}A = I$, where I is the identity or unit matrix.

The inverse of a 2×2 matrix:

Take for example a arbitury 2×2 Matrix A whose determinant (ad − bc) is not equal to zero.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a,b,c,d are numbers, The inverse is:

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The inverse of a n×n matrix:

The inverse of a n×n matrix A can be determined by using the relation

$$A^{-1} = \frac{adj(A)}{\det(A)}$$

where the adj(A) shows the adjoint of a matrix A. It can be determined by the following method:

- Given the n×n matrix A, define

  $B = b_{ij}$

  as a matrix whose coefficients are obtained by taking the determinant of the (n-1) × (n-1) matrix, taken by deleting the i^{th} row and j^{th} column of A. The terms of B are called the cofactors of A.

- Define the matrix C, where

  $c_{ij} = (-1)^{i+j}\, b_{ij}$.
- The transpose of matrix C i.e. C^T is said to be the adjoint of matrix A.

Now to find the inverse of A divide the matrix C^T by the determinant of A to get its inverse.

Example: If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then find inverse of matrix A .

Solution: We have $|A| = 1\,(16 - 9) - 3\,(4 - 3) + 3\,(3 - 4) = 1 \neq 0$

Here $A^{-1} = \dfrac{adj(A)}{|A|}$

$$= \frac{(cofactor\ matrix\ of\ A)^T}{|A|}$$

Now, we get

$$adj.(A) == \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore (A)^{-1} = \frac{(adj.A)}{|A|} = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Solving Systems of Equations using Matrices

Let, we have a system of linear equations with n equations and n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned}$$

The unknowns are shown by $x_1, x_2, \dots, x_n$ and the coefficients (a and b above) are assume to be given. In matrix form the system of equations is given by:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

or $\underline{A}x = \underline{b}$

Method of Solutions

(1) Inverse Matrix Method

(2) Cramer's Rule

(1) Inverse Matrix Method:

In inverse matrix method we use the inverse of a matrix to solve a system of equations, e. g. $Ax = b$. On pre-multiplication both sides of this equation by A^{-1} , we get

$$A^{-1}(Ax) = A^{-1}b$$
$$(A^{-1}A)x = A^{-1}b$$

or alternatively

$$x = A^{-1}b$$

Therefore by calculating the inverse of the matrix and multiplying this with the vector b we can obtain the solution of equations directly. Previously we have found that the inverse by

$$A^{-1} = \frac{adj(A)}{|A|}$$

It is clear that the existence of a solution depends on the determinant of A. There are three cases:

(i) If the det(A) does not equal zero then solutions exist using $x = A^{-1}b$

(ii) If the det(A) is zero and b=0 then the solution is not unique or it does not exist.

(iii)If the det(A) is zero and b=0 then the solution can be x = 0 but as with 2. is not unique or does not exist.

For two simultaneous equations:

$$ax + by = c$$
$$dx + ey = f$$

Written in matrix form

$$\begin{pmatrix} a & b \\ d & e \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ f \end{pmatrix}$$

The solution is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ d & e \end{pmatrix}^{-1}\begin{pmatrix} c \\ f \end{pmatrix}$$

where inverse of a matrix can be determined by

$$A^{-1} = \frac{adj(A)}{|A|}$$

Similarly for three simultaneous equations, let the equations be

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

In matrix form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

and by rearranging we would get that the solution would look like

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

where inverse of a matrix can be determined by

$$A^{-1} = \frac{adj(A)}{|A|}$$

(2) Cramer's Rule

In Cramer's rule we use determinants to solve systems of equations. Let the equations as,

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

The first term x_1 above can be found by replacing the first column of A by RHS. Doing this we obtain:

$$x_1 = \frac{1}{|A|}\begin{vmatrix} b_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ b_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

In the general case for solving x_p we replace the p^{th} column of matrix A by RHS and then expand the determinant. This method can be applied to solve systems of linear equations. We take this method for solving two simultaneous equations in x and y, and three equations with x, y and z.

Two simultaneous equations in x and y

$$ax + by = p$$
$$cx + dy = q$$

To solve these , we use

$$x = \frac{\begin{vmatrix} p & b \\ q & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \; y = \frac{\begin{vmatrix} a & p \\ c & q \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

Chapter 2. Straight Line

Straight Line

A line is simply an object in geometry that is characterized under zero width object that extends on both sides. A straight line is just a line with no curves. So, a line that extends to both sides till infinity and has no curves is called a straight line.

Equation of Straight Line

The relation between variables x, y satisfy all points on the curve.

(1) General Form of Straight Line

The general form of straight line is as given below:

$ax + by + c = 0$

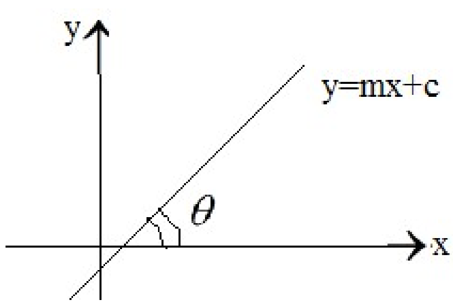
where x, y are variables and a,b, c are constants.

(2) Slope Form of Straight Line

The slope form of straight line is as given below:

$y=mx+c$

where m is called slope and c is the y-intercept.



(3) One Point Form of Straight Line

The One Point Form of straight line is as given below:

$$(y - y_1) = m(x - x_1)$$

where m is the slope and (x_1, y_1) is the point through which the line passes.

(4) Two Point Form of Straight Line

The equation of straight line through the two points (x_1, y_1) and (x_2, y_2) is given by

$$(y - y_1) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

(5) Intercept Form of Straight Line

x/a + y/b = 1

where a and b are x- and y-intercepts made by the **straight line**

Example : The equation of a line is given by, 2x – 6y +3 = 0. Find the slope and both the intercepts.

Solution:

The given equation 2x – 6y + 3 = 0 can be represented in slope-intercept form as:

y = x/3 + 1/2

Comparing it with y = mx + c,
Slope of the line, m = 1/3

Also, the above equation can be re-framed in intercept form as;

x/a + y/b = 1

2x – 6y = -3

x/(-3/2) – y/(-1/2) = 1

Thus, x-intercept is given as a = -3/2 and y-intercept as b = 1/2.

Example: Find the equation of the line that passes through the points (-2, 4) and (1, 2).

Solution:

We know general equation of a line passing through two points is:

y = mx + b

Here m = (2-4)/(1-(-2)) = -2/3

We can find the equation (by solving first for "b") if we have a point and the slope. So we need to choose one of the points and use it to solve for b. Using the point (–2, 4), we get:

4 = (– 2/3)(–2) + b

4 = 4/3 + b

4 – 4/3 = b

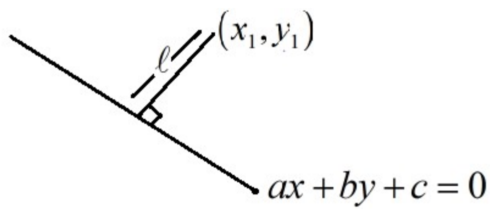
b = 8/3

so, y = (– 2/3) x + 8/3.

Which is the required equation of the line.

Length of Perpendicular from a Point on a Line

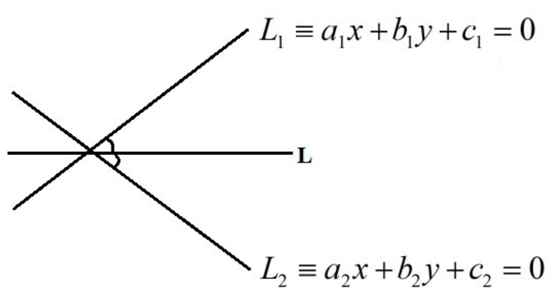
The length of the perpendicular from $P(x_1, y_1)$ on $ax + by + c = 0$ is



$$l = \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

Angular Bisector of Straight lines

An angle bisector has equal perpendicular distance from the two given lines. To find the equation of the bisector of the angle between lines, we have



The equation of line L can be given

$$\frac{a_1 x + b_1 y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2 x + b_2 y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

Example 1 Find the equation of angle bisectors for the pair of lines 3x + 4y – 7 = 0 and 4x + 3y – 7 = 0.

Solution All we have to do is apply the formula (derived here).

The required equations are

$$\frac{3x + 4y - 7}{\sqrt{9 + 16}} = \pm \frac{4x + 3y - 7}{\sqrt{16 + 9}}$$

On rearranging the terms, we get two bisectors as

x – y = 0

x + y = 2

Note that the two bisectors are perpendicular (and always will be).

We can find a little more information about these bisectors. By finding out the angle between (any one of the) bisectors and one of the lines, we can distinguish between the acute angle bisector and the obtuse angle bisector.

That is, if you find out the angle (θ) between the bisector x – y = 0, and one of the lines, 3x + 4y – 7 = 0, we get

|tanθ| = 7

This being greater than 1 indicates that θ is greater than 45°, making x – y = 0 the bisector of the obtuse angle between the given lines (and the other one of the acute angle).

Chapter 3. Circle

Circle Definition

A circle is a closed two-dimensional figure in which the set of all the points in the plane is equidistant from a given point called "centre". Every line that passes through the circle forms the line of reflection symmetry. Also, it has rotational symmetry around the centre for every angle. The circle formula in the plane is given as:

$$(x-h)^2 + (y-k)^2 = r^2$$

where (x,y) are the coordinate points (h,k) is the coordinate of the centre of a circle and r is the radius of a circle.

Properties of Circles

The important basic properties of circles are as follows:

- The outer line of a circle is at equidistant from the centre.
- The diameter of the circle divides it into two equal parts.
- Circles which have equal radii are congruent to each other.
- Circles which are different in size or having different radii are similar.
- The diameter of the circle is the largest chord and is double the radius.

Equation of a Circle in the Central Form:

The equation of the circle with centre (h, k) and the radius 'a' is,

$$(x-h)^2+(y-k)^2 = a^2$$

Note: The equation of a circle, with the centre as the origin is,

$$x^2+y^2= a^2$$

Equation of a Circle in General Form

The general equation of any type of circle is represented by:

$x^2 + y^2 + 2gx + 2fy + c = 0$, for all values of g, f and c.

Example :

Consider a circle whose centre is at the origin and radius is equal to 8 units.

Solution:

Given: Centre is (0, 0), radius is 8 units.

We know that the equation of a circle when the centre is origin:

$\mathbf{x^2 + y^2 = a^2}$

For the given condition, the equation of a circle is given as

$x^2 + y^2 = 8^2$

$x^2 + y^2 = 64$, which is the equation of a circle

Example :

Equation of a circle is $x^2+y^2-12x-16y+19=0$. Find the centre and radius of the circle.

Solution:

Given equation is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$,

$2g = -12$, $2f = -16$, $c = 19$

$g = -6$, $f = -8$

Centre of the circle is (6,8)

Radius of the circle = $\sqrt{[(-6)^2 + (-8)^2 - 19]} = \sqrt{[100 - 19]} =$

$= \sqrt{81} = 9$ units.

Therefore, the radius of the circle is 9 units.

Equation of a Circle in Diameter Form:

Let the coordinates of end points of a diameter be (x_1, y_1) and (x_2, y_2) then equation of circle in the diameter form is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Example: find the equation of a circle whose end points of a diameter be (2, 3) and (4, 8).

Solution: The equation of circle in the diameter form is given by

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Here points $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (4, 8)$

(x-2)(x-4)+(y-3)(y-8)=0

$$x^2 - 6x + 8 + y^2 - 11y + 24 = 0$$

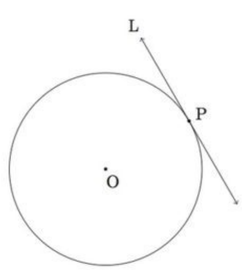
$$x^2 + y^2 - 6x - 11y + 32 = 0$$

Tangents to a Circle:

Tangent to a circle is a line that touches the circle at one point, which is known as Tangency. At the point of Tangency, Tangent to a circle is always perpendicular to the radius.

Tangent to a Circle

The line that joins two infinitely close points from a point on the circle is a **Tangent**. In other words, we can say that the lines that intersect the circles exactly in one single point are Tangents. Point of tangency is the point where the tangent touches the circle. At the point of tangency, a tangent is perpendicular to the radius. Several theorems are related to this because it plays a significant role in geometrical constructions and proofs. We will look at them one by one.



Properties of Tangents

Remember the following points about the properties of tangents-

- The tangent line never crosses the circle, it just touches the circle.
- At the point of tangency, it is perpendicular to the radius.
- A chord and tangent form an angle and this angle is same as that of tangent inscribed on the opposite side of the chord.
- From the same external point, the tangent segments to a circle are equal.

General Equation

Here, the list of the tangent to the circle equation is given below:

- The tangent to a circle equation $x^2 + y^2 = a^2$ at (x_1, y_1) is $\mathbf{xx_1 + yy_1 = a^2}$
- The tangent to a circle equation $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is $\mathbf{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0}$
- The tangent to a circle equation $x^2 + y^2 = a^2$ at $(a \cos \theta, a \sin \theta)$ is $\mathbf{x \cos \theta + y \sin \theta = a}$
- The tangent to a circle equation $x^2 + y^2 = a^2$ for a line $y = mx + c$ is $\mathbf{y = mx \pm a \sqrt{[1 + m^2]}}$

Condition of Tangency

The tangent is considered only when it touches a curve at a single point or else it is said to be simply a line.

The condition of tangency for a straight line $ax+by+c=0$ to a circle $x^2 + y^2 = r^2$ is

Radius = Perpendicular distance from the centre of circle to the straight line $ax+by+c=0$

$$r = \frac{a.0 + b.0 + c}{\sqrt{a^2 + b^2}}$$

Chapter 4. Limits and continuity

The concept of Limits and continuity is one of the most crucial topics in calculus.

Limit Definition

A limit of a function is a number that a function reaches as the independent variable of the function reaches a given value. The value (say a) to which the function f(x) gets close arbitrarily as the value of the independent variable x becomes close arbitrarily to a given value a symbolized as f(x) = A.

Points to remember :

- If $\lim_{x \to a-} f(x)$ is the expected value of f at x = a given the values of 'f' near x to the left of a. This value is known as **left-hand limit** of 'f' at a.
- If $\lim_{x \to a+} f(x)$ is the expected value of f at x = a given the values of 'f' near x to the right of a. This value is known as the **right-hand limit** of f(x) at a.
- If the right-hand and left-hand limits coincide, we say the common value as the limit of f(x) at x = a and denote it by $\lim_{x \to a} f(x)$.

One-Sided Limit

The limit that is based completely on the values of a function taken at x -value that is slightly greater or less than a particular value. A two-sided limit $\lim_{x \to a} f(x)$ takes the values of x into account that are both larger than and smaller than *a*. A one-sided limit from the left $\lim_{x \to a-} f(x)$ or from the right $\lim_{x \to a+} f(x)$ takes only values of x smaller or greater than *a* respectively.

Properties of Limit

- The limit of a function is represented as f(x) reaches L as x tends to limit a, such that; $\lim_{x \to a} f(x) = L$

- The limit of the sum of two functions is equal to the sum of their limits, such that: $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} [f(x)] + \lim_{x \to a} [g(x)]$

- The limit of any constant function is a constant term, such that, $\lim_{x \to a} C = C$

- The limit of product of the constant and function is equal to the product of constant and the limit of the function, such that: $\lim_{x \to a} [m.f(x)] = m \lim_{x \to a} [f(x)$

- Quotient Rule: $\lim_{x \to a} [f(x)/g(x)] = \lim_{x \to a} [f(x)] / \lim_{x \to a} [g(x)], \quad \lim_{x \to a} [f(x)] \neq 0$

Example: Find

$$\lim_{x \to -2} \left(3x^2 + 5x - 9\right)$$
$$= 3(4) + 5(-2) - 9$$
$$= 12 - 10 - 9$$
$$= -7$$

Example: Find $\lim_{x \to 3}$ [x(x+2)].

Solution: $\lim_{x \to 3}$ [x(x+2)] = 3(3+2) = 3 x 5 = 15

Example: Find the limit of

$$\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$$

Solution:

$$\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$$

$$= \lim_{x \to 5} \frac{(x-5)(x+5)}{x-5}$$

$$= \lim_{x \to 5} (x+5)$$

$$= 10$$

Continuity

A function is said to be continuous at a particular point if the following three conditions are satisfied.

1. f(a) is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = f(a)$

A function is said to be continuous if you can trace its graph without lifting the pen from the paper. But a function is said to be discontinuous when it has any gap in between. Let us see the types discontinuities.

Types of Discontinuity

There are basically two types of discontinuity:

- Infinite Discontinuity

- Jump Discontinuity

Infinite Discontinuity

A branch of discontinuity wherein, a vertical asymptote is present at x = a and f(a) is not defined. This is also called as Asymptotic Discontinuities. If a function has values on both sides of an asymptote, then it cannot be connected, so it is discontinuous at the asymptote.

Jump Discontinuity

A branch of discontinuity wherein limx→a+f(x)≠limx→a−f(x), but both the limits are finite. This is also called simple discontinuity or continuities of first kind.

Positive Discontinuity

A branch of discontinuity wherein a function has a pre-defined two-sided limit at x=a, but either f(x) is undefined at *a,* or its value is not equal to the limit at *a*.

Example: check the continuity of function $\dfrac{6x^3+5}{x-5}$

Solution: Since the function is not defined at x=5, therefore the function is discontinuous at x=5.

Example: Let a function be defined as $f(x)=\begin{cases} 5-2x \ \ for \ x<1 \\ 3 \ \ for \ x=1 \\ x+2 \ \ for \ x>1 \end{cases}$

check the continuity of function at x=1.

Solution

Now for x = 1, let us check all the three conditions of continuity:

Value of function: f(1) = 3

Left-Hand Limit:

$$\lim_{x \to 1-} f(x) = \lim_{h \to 0} f(1-h)$$
$$= \lim_{h \to 0} [5 - 2(1-h)]$$
$$= 5 - 2$$
$$= 3$$

Right-Hand Limit:–

$$\lim_{x \to 1+} f(x) = \lim_{h \to 0} f(1+h)$$
$$= \lim_{h \to 0} [5 - 2(1+h)]$$
$$= 5 - 2$$
$$= 3$$

Since Left-Hand Limit = Right-Hand Limit = f(1)

Therefore the function f(x) is continuous at x = 1.

Chapter 5. Differentiation

In calculus, differentiation is one of the two important concepts apart from integration. Differentiation is a method of finding the derivative of a function. It is used to find the instantaneous rate of change in function based on one of its variables.

If x is a variable and y is another variable, then the rate of change of x with respect to y is given by dy/dx. This is the general expression of derivative of a function and is represented as f'(x) = dy/dx, where y = f(x) is any function.

Basic Formulae of Differentiation

1. $\frac{d}{dx}\left(x^{n}\right)=nx^{n-1}$

2. $\frac{d}{dx}\left(x\right)=1$

3. $\frac{d}{dx}\left(a^{x}\right)=a^{x}\log_{e}a$

4. $\frac{d}{dx}\left(e^{x}\right)=e^{x}$

5. $\frac{d}{dx}\left(\log x\right)=\frac{1}{x}$

6. $\frac{d}{dx}\left(c\right)=0$, where c is a constant

7. $\frac{d}{dx}\left(\sin x\right)=\cos x$

8. $\frac{d}{dx}\left(\cos x\right)=-\sin x$

9. $\frac{d}{dx}\left(\tan x\right)=\sec^{2}x$

10. $\frac{d}{dx}(\cot x) = -\cos ec^2 x$

11. $\frac{d}{dx}(\sec x) = \sec x.\tan x$

12. $\frac{d}{dx}(co\sec x) = -\cot x.\cos ecx$

13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

14. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

15. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

16. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

Differentiation of Sum and Difference of Two Functions

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

Example. Find the derivative of $\left(\sin(x) + e^x\right)$

Solution.

$$\begin{aligned}\frac{d}{dx}\left(\sin(x) + e^x\right) &= \frac{d}{dx}\sin(x) + \frac{d}{dx}e^x \\ &= \cos x + e^x\end{aligned}$$

Example. Find the derivative of $x^2 + \sin^{-1} x - \log 5x$

Solution.

$$\frac{d}{dx}\left(x^2 + \sin^{-1} x - \log 5x\right) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin^{-1} x) - \frac{d}{dx}(\log 5x)$$

$$2x + \frac{1}{1-x^2} - \frac{1}{5x}$$

Differentiation of Product of Two Functions

$$\frac{d}{dx}\left[f(x).g(x)\right] = \frac{d}{dx}\left[f(x)\right].\left[g(x)\right] + \left[f(x)\right].\frac{d}{dx}\left[g(x)\right]$$

Example. Find the derivative of $x^4 \log 8x$

Solution.

$$\frac{d}{dx}\left[x^4 \log 8x\right] = \frac{d}{dx}\left[x^4\right]\log 8x + x^4 \frac{d}{dx}[\log 8x]$$
$$= 4x^3.\log 8x + x^4.\frac{8}{8x}$$
$$= 4x^3.\log 8x + x^3$$

Differentiation of Division of Two Functions

$$\frac{d}{dx}\left[f(x)/g(x)\right] = \frac{\frac{d}{dx}\left[f(x)\right].\left[g(x)\right] - \left[f(x)\right].\frac{d}{dx}\left[g(x)\right]}{\left[g(x)\right]^2}$$

Example. Find the derivative of $\dfrac{x^8}{\sin 2x}$

Solution.

$$\frac{d}{dx}\left[\frac{x^8}{\sin 2x}\right]=\frac{\sin 2x.8x^7 - x^8.2\cos 2x}{[\sin 2x]^2}$$

$$=2\left[\frac{4\sin 2x.x^7 - x^8.\cos 2x}{[\sin 2x]^2}\right]$$

Example. Find the derivative of $\frac{e^x}{\log 2x}$

Solution.

$$\frac{d}{dx}\left[\frac{e^x}{(\log 2x)}\right]=\frac{\log 2x.e^x - e^x.\frac{2}{2x}}{(\log 2x)^2}$$

$$=\frac{x\log 2x.e^x - e^x}{x(\log 2x)^2}$$

Chapter 6. Maxima and Minima

As the name suggests, this topic is devoted to the method of finding the maximum and the minimum values of a function in a given domain. It finds application in almost every field of work, and in every subject. Let's find out more about the maxima and minima in this topic.

Types of Maxima and Minima

The maxima or minima can also be called an extremum i.e. an extreme value of the function. Let us have a function $y = f(x)$ defined on a known domain of x. Based on the interval of x, on which the function attains an extremum, the extremum can be termed as a 'local' or a 'global' extremum. Let's understand it better in the case of maxima.

Local Maxima

A point is known as a Local Maxima of a function when there may be some other point in the domain of the function for which the value of the function is more than the value of the local maxima, but such a point doesn't exist in the vicinity or neighborhood of the local maxima. You can also understand it as a maximum value with respect to the points nearby it.

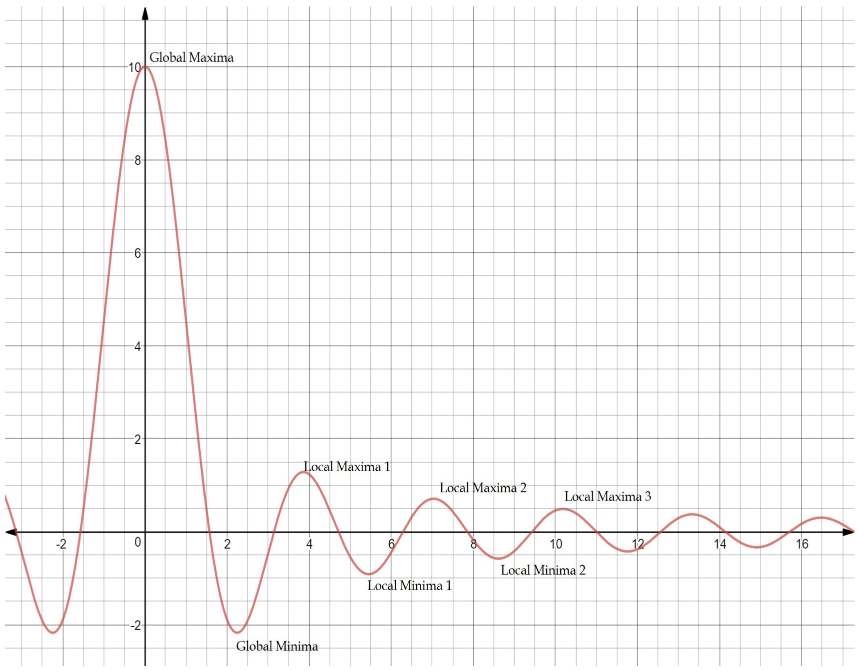
Global Maxima

A point is known as a Global Maxima of a function when there is no other point in the domain of the function for which the value of the function is more than the value of the global maxima. Types of Global Maxima:

Global maxima may satisfy all the conditions of local maxima. You can also understand it as the Local Maxima with the maximum value in this case.

Alternately, the global maxima for an increasing function could be the endpoint in its domain; as it would obviously have the maximum value. In this case, it isn't a local maximum for the function.

Similarly, the local and the global minima can be defined. Look at the graph below to identify the different types of maxima and minima.



The Second Derivative Test

This test is used to determine whether a stationary point is a Local Maxima or a Local Minima. Whether it is a global maxima/global minima can be determined by comparing its value with other local maxima/minima. Let us have a function y = f(x) with $x = x_0$ as a stationary point. Then the test says:

- If $\left[\frac{d^2 f}{dx^2}\right] > 0$, then $x = x_0$ is a point of Local Minima.
- If $\left[\frac{d^2 f}{dx^2}\right] < 0$, then $x = x_0$ is a point of Local Maxima.

Question 1 : Find the local maxima and minima for the function $y = x^3 - 3x + 2$.

Answer : First we have to find the stationary points for this function.

$y = x^3 - 3x + 2$

$\therefore \frac{dy}{dx} = 3x^2 - 3$

put $\frac{dy}{dx} = 0$

$\therefore 3x^2 - 3 = 0$

$\Rightarrow x^2 - 1 = 0$

$\Rightarrow x = \pm 1$

Also $\frac{d^2y}{dx^2} = 6x$

Therefore, stationary points are x=1, x= -1

Now $\left(\frac{d^2y}{dx^2}\right)_{x=1} = 6$ and $\left(\frac{d^2y}{dx^2}\right)_{x=-1} = -6$

Therefore x=1is a point of Local Minima and x = -1 is a point of Local Maxima.

Example: A ball is thrown in the air. Its height at any time t is given by:

$$h = 3 + 14t - 5t^2$$

What is its maximum height?

Using derivatives we can find the slope of that function:

$\frac{dh}{dt} = 14 - 10t$

Now find when the **slope is zero**:

14 − 10t = 0

10t = 14

t = 14 / 10 = **1.4**

The slope is zero at **t = 1.4 seconds**

And the height at that time is:

$h = 3 + 14 \times 1.4 - 5 \times 1.4^2$

$h = 3 + 19.6 - 9.8 = \mathbf{12.8}$

And so:

The maximum height is **12.8 m** (at t = 1.4 s)

Chapter 7. Integration

Integration is the calculation of an integral. Integrals in maths are used to find many useful quantities such as areas, volumes, displacement, etc. When we speak about integrals, it is related to usually definite integrals. The indefinite integrals are used for antiderivatives.

Integration Definition

The integration denotes the summation of discrete data. The integral is calculated to find the functions which will describe the area, displacement, volume, that occurs due to a collection of small data, which cannot be measured singularly. In a broad sense, in calculus, the idea of limit is used where algebra and geometry are implemented.

"Integral is based on a limiting procedure which approximates the area of a curvilinear region by breaking the region into thin vertical slabs." Learn more about Integral calculus here.

Definite Integral

An integral that contains the upper and lower limits then it is a definite integral. On a real line, x is restricted to lie. Riemann Integral is the other name of the Definite Integral.

A definite Integral is represented as: $\int_a^b f(x)dx$

Indefinite Integral

Indefinite integrals are defined without upper and lower limits. It is represented as:

Let $\frac{d[F(x)]}{dx} = f(x)$. Then, $\int f(x)dx = F(x) + c$ we write . These integrals are called indefinite integrals or general integrals, C is called a constant of integration. All these integrals differ by a constant.

If two functions differ by a constant, they have the same derivative.

Geometrically, the statement $\int f(x)dx = F(x) + c = y$ (say) represents a family of curves. The different values of C correspond to different members of this family and these members can be obtained by shifting any one of the curves parallel to itself. Further, the tangents to the curves at the points of intersection of a line $x = a$ with the curves are parallel.

Integration Formulas

Check below the formulas of integral or integration, which are commonly used in higher-level maths calculations. Using these formulas, you can easily solve any problems related to integration.

1. $\int x^n dx = \frac{\left(x^{n+1}\right)}{\left(x^{n+1}\right)} + c, \ \ n \neq -1$

2. $\int dx = x + c$

3. $\int a^x dx = \frac{a^x}{\log_e a} + c, \ \ a > 0, \ a \neq 1$

4. $\int \left(e^x\right) dx = e^x + c$

5. $\int \frac{1}{x} dx = \left(\log x\right) + c$

6. $\int \cos x dx = \left(\sin x\right) + c$

7. $\int \sin x dx = -\left(\cos x\right) + c$

8. $\int \sec^2 x dx = \left(\tan x\right) + c$

9. $\int \cos ec^2 x dx = -\left(\cot x\right) + c$

10. $\int \sec x . \tan x dx = \left(\sec x\right) + c$

11. $\int \cot x . \cos ecx dx = -\left(co \sec x\right) + c$

12. $\int \frac{1}{\sqrt{1-x^2}}dx = \left(\sin^{-1}x\right) + c$

13. $\int \frac{1}{\sqrt{1-x^2}}dx = -\left(\cos^{-1}x\right) + c$

14. $\int \frac{1}{1+x^2}dx = \left(\tan^{-1}x\right) + c$

15. $\int \frac{1}{x\sqrt{x^2-1}}dx = \left(\sec^{-1}x\right) + c$

Differentiation of Sum and Difference of Two Functions

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$$

Properties of indefinite integrals

(i) The process of differentiation and integration are inverse of each other,

i.e., $$\begin{aligned} &\int f(x)dx = F(x) \\ &\Rightarrow \frac{d[F(x)}{dx} = f(x) + c \end{aligned}$$

where C is any arbitrary constant.

(ii) Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent. So if f and g are two functions such that

$\frac{d}{dx}\int f(x)dx = \frac{d}{dx}\int g(x)dx$, then $\int f(x)dx$ and $\int g(x)dx$ are equivalent.

(iii) The integral of the sum of two functions equals the sum of the integrals of

the functions i.e., $\int [f(x)+g(x)]dx = \int [f(x)]dx + \int [g(x)]dx$

(iv) A constant factor may be written either before or after the integral sign, i.e.,

$\int af(x)dx = a\int f(x)dx$, where 'a' is a constant.

Methods of Integration

There are some methods or techniques for finding the integral where we can not directly select the antiderivative of function *f* by reducing them into standard forms.

Some of these methods are based on

1. Integration by substitution
2. Integration using partial fractions
3. Integration by parts.

Example: Find the integral $\int (\sin x - e^x - \sec x \tan x)dx$

Solution: $\int (\sin x - e^x - \sec x \tan x)dx$

$$= \int \sin x dx - \int e^x dx - \int \sec x \tan x dx$$
$$= -\cos x - e^x - \sec x + c$$

Example: Find the integral $\int \frac{dx}{x^2 - a^2}$

Solution:

$\int \frac{dx}{x^2 - a^2} = \int \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx$ (By Partial Fraction)

$$= \int \frac{1}{2a}\left(\frac{1}{x-a}\right)dx - \int \frac{1}{2a}\left(\frac{1}{x+a}\right)dx$$
$$= \frac{1}{2a}\log(x-a) - \frac{1}{2a}\log(x+a) + c$$
$$= \frac{1}{2a}\log\frac{x-a}{x+a} + c$$

Example: Find the integral $\int 2x \sin\left(x^2 + 1\right)dx$

Solution:

Put $x^2+1=t$ (Substitution)

$$\Rightarrow 2xdx = dt$$

$$\therefore \int 2x\sin\left(x^2+1\right)dx = \int t\sin t$$

$$= t(-\cos t) - \int 1.(-\cos t)dt \text{ (Integration by parts)}$$

$$= -t\cos t + \sin t + c$$

$$= -(x^2+1)\cos(x^2+1) + \sin(x^2+1) + c$$

Definite integral

The definite integral is denoted by $\int_A^B f(x)dx$, where A is the lower limit of the integral and B is the upper limit of the integral. The definite integral is evaluated in the following two ways:

(i) The definite integral as the limit of the sum

(ii) $\int_A^B f(x)dx = F(B) - F(A)$

if F is an antiderivative of $f(x)$.

Fundamental Theorem of Calculus

(i) **Area function:** The function A (x) denotes the area function and is given

by A (x) = $\int_a^b f(x)dx$

(ii) **First Fundamental Theorem of integral Calculus**

Let f be a continuous function on the closed interval [a, b] and let A (x) be

the area function . Then $A'(x) = f(x)$ for all $x \in \left[a,b\right]$.

(iii) **Second Fundamental Theorem of Integral Calculus**

Let f be continuous function defined on the closed interval [a, b] and F be

an antiderivative of f.

$$\int_a^b f(x)dx = F(x)_a^b = F(B) - F(A)$$

Some properties of Definite Integrals

(i) $\int_a^b f(x)dx = \int_a^b f(t)dt$

(ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

(iv) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(v) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

(vi) $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a (x)dx, & f(2a-x) = f(x) \\ 0, & f(2a-x) = -f(x) \end{cases}$

(vii) $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a (x)dx, & f(x)\text{ is even function} \\ 0, & f(x)\text{ is odd function} \end{cases}$

Example: Find the integral $\int_1^2 (x^2 + 2)dx$

Solution:

$$\int_1^2 (x^2+2)dx = \left(\frac{x^3}{3}+2x\right)_{x=1}^{x=2}$$

$$= \left(\frac{8}{3}+4\right)-\left(\frac{1}{3}+2\right)$$

$$= \frac{7}{3}-2$$

$$= \frac{1}{3}$$

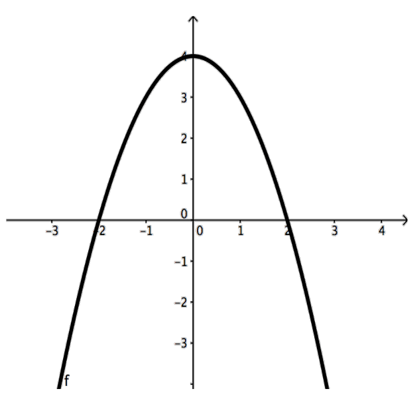
Finding the Area with Integration

Finding the area of space from the curve of a function to an axis on the Cartesian plane is a fundamental component in calculus. Definite integration finds the accumulation of quantities, which has become a basic tool in calculus and has numerous applications in science and engineering. While it is used to make formulas in physics more comprehensible, often it is used to optimize the use of space in a given area.

Definite Integration

Whenever we are calculating area in a given interval, we are using definite integration. Lets try to find the area under a function for a given interval.

(1) Integrate $f(x) = -x^2 + 4$ from [-2, 2].



Taking the integral.

$$\int_a^b f(x)dx = \int_{-2}^{2} (-x^2+4)dx$$

Find the Integral.

$$\int_{-2}^{2}(-x^2+4)dx = -\frac{1}{3}x^3 + 4x$$

Integrate from the given interval, [-2,2].

$$\int_{-2}^{2} f(x)\,dx = F(2) - F(-2)$$

$$\int_{-2}^{2} f(x)dx = [-\frac{1}{3}(2)^3 + 4(2)] - [-\frac{1}{3}(-2)^3 + 4(-2)]$$

$$= [-\frac{8}{3} + 8] - [\frac{16}{3} - 8]$$

=32/3

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